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Finite size giant magnons in the string dual of $\mathcal{N} = 6$ superconformal Chern-Simons theory

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Troels Harmark,[a] **Gianluca Grignani,**[b] **Marta Orselli**[a] **Gordon W. Semenoff**[c]

[a] *The Niels Bohr Institute, Blegdamsvej 17, 2100 Copenhagen Ø, Denmark*
[b] *Dipartimento di Fisica, Università di Perugia and I.N.F.N. Sezione di Perugia, Via Pascoli, I-06123 Perugia, Italy*
[c] *Department of Physics and Astronomy, University of British Columbia, Vancouver, British Columbia, V6T 1Z1 Canada*
E-mail: harmark@nbi.dk, grignani@pg.infn.it, orselli@nbi.dk, gordonws@phas.ubc.ca

Abstract: We find the exact solution for a finite size Giant Magnon in the SU(2) $\times$ SU(2) sector of the string dual of the $\mathcal{N} = 6$ superconformal Chern-Simons theory recently constructed by Aharony, Bergman, Jafferis and Maldacena. The finite size Giant Magnon solution consists of two magnons, one in each SU(2). In the infinite size limit this solution corresponds to the Giant Magnon solution of arXiv:0806.4959. The magnon dispersion relation exhibits finite-size exponential corrections with respect to the infinite size limit solution.

Keywords: AdS-CFT Correspondence, Bosonic Strings.

Contents

1. Introduction and summary

Recently, motivated by the possible description of the worldvolume dynamics of coincident membranes in M-theory, a new class of conformal invariant, maximally supersymmetric field theories in 2+1 dimensions has been found [1, 2]. These theories contain gauge fields with Chern-Simons-like kinetic terms. Based on this development, Aharony, Bergman, Jafferis and Maldacena proposed a new gauge/string duality between an $\mathcal{N} = 6$ super-conformal Chern-Simons theory (ABJM theory) and type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{C}P^3$ [2]. This is conjectured to constitute a new exact duality between gauge and string theory in addition to the celebrated duality between $\mathcal{N} = 4$ superconformal Yang-Mills (SYM) theory and type IIB string theory on $\mathrm{AdS}_5 \times S^5$.

The ABJM theory consists of two Chern-Simons theories of level k and $-k$ and each with gauge group $\mathrm{U}(N)$. It has two pairs of chiral superfields transforming in the bifundamental representations of $\mathrm{U}(N) \times \mathrm{U}(N)$. The R-symmetry is SU(4) in accordance with the $\mathcal{N} = 6$ supersymmetry of the theory. It was observed in [2] that one can define a 't Hooft coupling $\lambda = N/k$. In the 't Hooft limit $N \to \infty$ with λ fixed one has a continuous coupling λ and the ABJM theory is weakly coupled for $\lambda \ll 1$. The ABJM theory is conjectured to be dual to M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ with N units of four-form flux which for $k \ll N \ll k^5$ can be compactified to type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{C}P^3$.

In the $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality major progress has been achieved in following the tantalizing idea that the planar limit of $\mathcal{N} = 4$ Yang-Mills theory and its string dual, the type IIB string theory on $AdS_5 \times S^5$ background, might be integrable models which could be completely solvable using a Bethe ansatz [3−5]. This brings naturally the hope that also the new $\mathrm{AdS}_4/\mathrm{CFT}_3$ duality can be solvable using a Bethe ansatz [6, 7]. However, as shown in [8] this could be a more challenging task than for $\mathrm{AdS}_5/\mathrm{CFT}_4$ since the magnon dispersion relation in the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ sector of ABJM theory is shown to contain a non-trivial function of λ, interpolating between weak and strong coupling. A fundamental consequence of having a Bethe ansatz is that it has distinct quasi-particles, the magnons.

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In [8] the question of the magnon dispersion relation was considered both from the point of view of a sigma-model limit, a Penrose limit (see also [9, 7]), and furthermore using a new Giant Magnon solution (see also [7]). All this was done in the $SU(2) \times SU(2)$ sector of ABJM theory, corresponding to two two-spheres S^2 in the $\mathbb{C}P^3$ space. Adding the weak coupling result of [6, 7] and assuming that the symmetry arguments of [10] also can be applied to the AdS_4/CFT_3 duality, the following dispersion relation was found [7, 8]

$$\Delta = \sqrt{\frac{1}{4} + h(\lambda) \sin^2\left(\frac{p}{2}\right)}, \quad h(\lambda) = \begin{cases} 4\lambda^2 + \mathcal{O}(\lambda^4) \text{ for } \lambda \ll 1 \\ 2\lambda + \mathcal{O}(\sqrt{\lambda}) \text{ for } \lambda \gg 1 \end{cases} \tag{1.1}$$

In this paper we investigate further the integrability of the AdS_4/CFT_3 correspondence by constructing a new finite size Giant Magnon solution for type IIA string theory on $AdS_4 \times \mathbb{C}P^3$.

The Giant Magnon solution found in [8] is a soliton of the world-sheet sigma model (living on $\mathbb{R} \times S^2 \times S^2$) whose image in spacetime is a string which is pointlike in AdS_4 and which rotate uniformly around the two S^2's with open endpoints moving at the speed of light on the equators of the two S^2's. The solution corresponds to one of the fundamental excitations of the spin chain with alternating sites between the fundamental and anti-fundamental representations of the gauge theory scalar fields and next to nearest neighbor interactions, found in [6, 7]. The string orientation on the two S^2 is opposite, so that the Giant Magnon solution can be interpreted as two giant magnons moving with equal momenta with the same polar angle and opposite azimuthal angle.

In the infinite volume limit, integrability implies scattering with a factorized S-matrix. The Bethe equation are then of the asymptotic type, but eventually an important problem that the integrability program would have to address is that of finite size corrections. In this paper we address this problem and derive exactly the conserved charges at finite size for the Giant Magnon of the type IIA string theory on $AdS_4 \times \mathbb{C}P^3$.

Finite size corrections to the Giant Magnon dispersion relation were first found by generalizing Hofman and Maldacena's Giant Magnon solution in the type-IIB sigma model on $AdS_5 \times S^5$ to the case where the size is finite [11, 12].[1] In [12] in particular it was shown that the finite size Giant Magnon becomes a physical string configuration, once defined on a Z_M orbifold of S^5. The quantization of this Giant Magnon away from the infinite size limit was discussed in [14] where it was argued that this quantization inevitably leads to string theory on a Z_M-orbifold of S^5. We shall show that also for the $AdS_4 \times \mathbb{C}P^3$ Giant Magnon it would be possible to identify the string endpoints by considering an orbifold of $\mathbb{C}P^3$ [15]. The orbifold identification makes of this a legitimate closed string solution, as in [12, 14] for the $AdS_5 \times S^5$ Giant Magnon.

Computing the magnon spectrum in an asymptotic expansion about infinite size, we find that the dispersion relation, up to the leading exponential correction, is

$$\Delta - J = 2\sqrt{2\lambda}\left|\sin\frac{p}{2}\right| - 8\sqrt{2\lambda}\left|\sin\frac{p}{2}\right|^3 e^{-2-J/(\sqrt{2\lambda}|\sin\frac{p}{2}|)} + \cdots \tag{1.2}$$

[1] See also [13] for the case of an arbitrary number of Giant Magnons.

where p is the magnon momentum on each of the two S^2. The finite size corrections are exponentially small with large $J = \frac{J_1 - J_3}{2}$, where J_1 and J_3 are the generators of the azimuthal translations on the two two-spheres.

Finite size correction to the magnon dispersion relation on $AdS_5 \times S^5$ have been reproduced from the gauge theory side using generalized Lüscher formulas for finite size corrections [16]. The result agrees with the classical string computation of [11, 12]. It would be extremely interesting to make the same comparison in the case of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ duality. However, the difference between the more standard $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality and the $\mathrm{AdS}_4/\mathrm{CFT}_3$ duality is that the latter only possesses 24 supersymmetries. Therefore the checks of $\mathrm{AdS}_4 \times \mathbb{C}P^3$ might indeed be more challenging than those for $\mathrm{AdS}_5/\mathrm{CFT}_4$.

The Hofman-Maldacena Giant Magnon is a $\frac{1}{2}$-BPS state and as such it has been shown to be part of a 16 dimensional short multiplet of the $\mathrm{SU}(2|2) \times \mathrm{SU}(2|2)$ symmetry [17, 14]. It would be very interesting to study and describe the supersymmetry properties of the newly found Giant Magnon solution [8] and of its finite size version derived here. An interesting question in fact is what happens to the supersymmetry in finite volume when a magnon is present. As argued in [14] an orbifold projection breaks at least half of the supersymmetry of the $AdS_5 \times S^5$ background. A string theory with a finite size Giant Magnon therefore cannot have the same number of supersymmetries of the parent type theory — and could even have no supersymmetry at all. As in [17] the study of the existence of fermion zero modes for the fermion fluctuations of the Green-Schwarz string sigma model on $\mathrm{AdS}_4/\mathrm{CFT}_3$ [18, 19] at infinite or at finite size, might shed some light on these questions.

2. The classical solution

To find the Giant Magnon solution on $AdS_4 \times \mathbb{C}P^3$ we consider the string sigma model on this background. The coordinates can be taken as a 5-vector Y and an 8-vector X where $X \in S^7$, $Y \in AdS_4$ constrained by

$$X^2 = \sum_{i=1}^{8} X_i X_i = 1\,, \quad Y^2 = \sum_{i=1}^{3} Y_i^2 - Y_4^2 - Y_5^2 = -1 \tag{2.1}$$

$$C_1 = \sum_{i=1,3,5,7} (X_i \partial_t X_{i+1} - X_{i+1} \partial_t X_i) = 0\,, \quad C_2 = \sum_{i=1,3,5,7} (X_i \partial_s X_{i+1} - X_{i+1} \partial_s X_i) = 0 \tag{2.2}$$

The constraints $C_1 = 0$ and $C_2 = 0$ define the background to be $\mathbb{C}P^3$.

The bosonic part of the sigma model action in the conformal gauge is

$$S = -\sqrt{2\lambda} \int dt \int ds \left[\frac{1}{4} \partial_a Y \cdot \partial^a Y + \partial_a X \cdot \partial^a X + \tilde{\Lambda}(Y^2 + 1) + \Lambda(X^2 - 1) + \Lambda_1 C_1^2 + \Lambda_2 C_2^2 \right] \tag{2.3}$$

The coupling constant of the sigma model is the inverse radius of curvature squared of the constant curvature spaces $\frac{1}{R^2} = \frac{1}{4\pi\sqrt{2\lambda}}$. The worldsheet metric has signature $(-,+)$ such that $\partial_a \partial^a = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial s^2} = -\partial_+ \partial_-$ and $\sigma^{\pm} = \frac{1}{2}(t \pm s)$. Here, Λ, $\tilde{\Lambda}$ and Λ_i, $i = 1, 2$ are Lagrange multipliers which enforce the coordinate constraints (2.1) and (2.2). In closed string theory, the worldsheet has cylindrical topology. The range of the time coordinate t

is infinite and the range of the space coordinate is taken to be $s \in (-r, r]$. The parameter r can be changed by scaling the worldsheet coordinates. We shall later fix r to a convenient value. The equations of motion following from the action (2.3) should be supplemented by Virasoro constraints,

$$\partial_+ X \cdot \partial_+ X + \frac{1}{4}\partial_+ Y \cdot \partial_+ Y = 0 \ , \quad \partial_- X \cdot \partial_- X + \frac{1}{4}\partial_- Y \cdot \partial_- Y = 0 \tag{2.4}$$

The Giant Magnon solution will be found as a solution of the classical equations of motion where only coordinates on two $S^2 \subset S^7$ and $R^1 \subset AdS_7$ are excited. The solution on $AdS_5 \times S^5$ was originally found by Hofman and Maldacena [20] in the limit where r is infinite. This is a closed string solution with open boundary conditions in one azimuthal direction.

In the case we are studying the solution is point-like in AdS_4 and extended along the two S^2 which are subsets of S^7. The solution lives on an $R^1 \times S^2 \times S^2$ subspace of $AdS_4 \times S^7$, the $R^1 \subset AdS_7$ and $S^2 \times S^2 \subset S^7$. We shall choose the solution in such a way that it has opposite azimuthal angles on the two S^2 and the same polar angles. All variables are periodic, except for the azimuthal angles of the two S^2's which will be chosen to obey the magnon boundary condition which on one S^2 is

$$\Delta\phi_1 \equiv p \tag{2.5}$$

and on the other one will be

$$\Delta\phi_2 \equiv -p \tag{2.6}$$

These identifications corresponds to opposite orientations of the string on the two S^2. The Giant Magnon is then characterized by the momentum p and by the choice of the point in the transverse directions to the two S^2, *i.e.* by 2 two-component polarization vectors. p has to be interpreted as the momentum of the magnons in the spin chain, these two magnons have equal magnon momentum. They give the same contribution to the total momentum constraint.

We begin with the ansatz that the solution lives on an $R^1 \times S^2 \times S^2$ subspace of $AdS_4 \times S^7$, with $\phi_1 = -\phi_2 = \phi$ and $\theta_1 = \theta_2 = \theta$, thus the ansatz is

$$Y_4 + iY_5 = e^{i\frac{\Delta}{r\sqrt{2\lambda}}t} \ , \qquad Y_1 = Y_2 = Y_3 = 0 \tag{2.7}$$

$$X_1 + iX_2 = \frac{1}{\sqrt{2}}e^{i\phi(t,s)}\sqrt{1 - z^2(t,s)}, \qquad X_5 + iX_6 = \frac{1}{\sqrt{2}}e^{-i\phi(t,s)}\sqrt{1 - z^2(t,s)}$$

$$(X_3, X_4) = \frac{\hat{n}_1}{\sqrt{2}}z(t,s) \ , \qquad (X_7, X_8) = \frac{\hat{n}_2}{\sqrt{2}}\, z(t,s) \tag{2.8}$$

where $\hat{n}_i$ $i = 1, 2$ are constant unit vectors and $z(t,s) = \cos\theta(t,s)$ is a function taking values on the interval $0 \le z < 1$. The ansatz (2.7)–(2.8) is such that the constraints (2.1) and (2.2) are automatically satisfied. The boundary condition for the magnon is

$$\phi(t, s = r) - \phi(t, s = -r) \ = \ p \tag{2.9}$$

with all other variables periodic.

Using (2.9) we see that the boundary conditions of the X_i variables at finite size are

$$X_1 + iX_2|_{s=r} = e^{ip}\left(X_1 + iX_2\right)|_{s=-r} \ , \qquad X_5 + iX_6|_{s=r} = e^{-ip}\left(X_5 + iX_6\right)|_{s=-r}$$

$$(X_3, X_4)|_{s=r} = (X_3, X_4)|_{s=-r} \ , \qquad (X_7, X_8)|_{s=r} = (X_7, X_8)|_{s=-r} \tag{2.10}$$

For $s \to \infty$ we get from (2.10) the boundary conditions of the solution found in [8]. The $\mathbb{C}P^3$ corresponds to making the identification

$$X_{2j-1} + iX_{2j} = Z(\hat{X}_{2j-1} + i\hat{X}_{2j}), \quad j = 1, \ldots, 4 \tag{2.11}$$

where $Z \in \mathbb{C}$ and X_i and $\hat{X}_i$ identify two points on $\mathbb{C}^4$. We see that the two string endpoints at $s = \pm r$ are *not* identified under this. Quite remarkably, however, it is possible to find a setting where the boundary conditions (2.10) correspond to identified endpoints. In ref. [15] orbifold projections of the ABJM theory were considered. These give non-chiral and chiral $(\mathrm{U}(N) \times \mathrm{U}(N))^n$ superconformal quiver gauge theories. These theories at level k are dual to certain $AdS_4 \times S^7/(Z_M \times Z_k)$ backgrounds of M-theory. In particular in ref. [15] an orbifold projection of the non-chiral ABJM theory, which produces a chiral gauge theory, was considered. This was done by placing N M_2-branes at the singularity of $\mathbb{C}^4/(Z_M \times Z_k)$, where the Z_M action is given by

$$\begin{aligned} X_1 + iX_2 &\to e^{\frac{2\pi i m}{M}} (X_1 + iX_2)\,, \qquad & X_5 + iX_6 &\to e^{-\frac{2\pi i m}{M}} (X_5 + iX_6) \\ X_3 + iX_4 &\to X_3 + iX_4\,, & X_7 + iX_8 &\to X_7 + iX_8 \end{aligned} \tag{2.12}$$

These identifications are identical to those produced by the Giant Magnon boundary conditions (2.10) if we chose $p = \frac{2\pi m}{M}$. We can thus conclude that, as for the $AdS_5 \times S^5$ finite size giant magnon [11, 12], considering an orbifold of the original theory [12], the endpoints of the string are identified, *i.e.* the orbifold group acts in such a way that it identifies the ends of the string, resulting in a legitimate state of closed string theory. This was advocated in [12] as a way to study the spectrum of a single magnon in a setting, $AdS_5 \times S^5/Z_M$, where it is a physical state and there are no issues of gauge invariance [11]. The same thing seems to happen also for the $AdS_4 \times \mathbb{C}P^3$ magnon, a natural setting for giving physical sense to this solution as a closed string state is to put it on an orbifold. We could then argue as in [14] that if we consider the giant magnon at finite size as a quantum string state, with the boundary condition that the string is open in the direction of the magnon motion, we are *inevitably* led to an orbifold.

The solution on AdS_4 in (2.7) is chosen so that the energy density is constant and the total energy of the string is the integral

$$\Delta = -\frac{\sqrt{2\lambda}}{2} \int_{-r}^{r} ds \left[Y_4 \dot{Y}_5 - Y_5 \dot{Y}_4 \right] \tag{2.13}$$

With the ansatz (2.7) and (2.8) the action reduces to

$$S = \sqrt{2\lambda} \int dt \int_{-r}^{r} ds \left[(1 - z^2)\partial_+\phi\partial_-\phi + \frac{\partial_+ z \partial_- z}{1 - z^2} \right] \tag{2.14}$$

up to a constant. The equations of motion are

$$\partial_+ \left((1 - z^2)\partial_-\phi \right) + \partial_- \left((1 - z^2)\partial_+\phi \right) = 0 \tag{2.15}$$

$$\partial_+ \left(\frac{\partial_- z}{1 - z^2} \right) + \partial_- \left(\frac{\partial_+ z}{1 - z^2} \right) = \frac{2z\partial_+ z\partial_- z}{(1 - z^2)^2} - 2z\partial_+\phi\partial_-\phi \tag{2.16}$$

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and the Virasoro constraints are

$$T_{++} = (1-z^2)\partial_+\phi\partial_+\phi + \frac{\partial_+ z\partial_+ z}{1-z^2} - \frac{1}{4}\left(\frac{\Delta}{r\sqrt{2\lambda}}\right)^2 \sim 0 \tag{2.17}$$

$$T_{--} = (1-z^2)\partial_-\phi\partial_-\phi + \frac{\partial_- z\partial_- z}{1-z^2} - \frac{1}{4}\left(\frac{\Delta}{r\sqrt{2\lambda}}\right)^2 \sim 0 \tag{2.18}$$

We will choose the parameter r so that

$$r = \frac{\Delta}{2\sqrt{2\lambda}} \tag{2.19}$$

this simplifies the constraints (2.17) and (2.18) so that the last term in each expression is equal to 1. In the standard solution r would not contain the factor of 1/2. It is easy to check that the Virasoro constraints are compatible with the equations of motion

$$\partial_- T_{++} = 0 \ , \ \ \partial_+ T_{--} = 0 \tag{2.20}$$

Now, we make the ansatz that the Giant Magnon on S^2 is a right-moving soliton

$$\phi(t,s) = \Psi t + \Omega s + \varphi(u) \ , \ \ z(t,s) = z(u) \tag{2.21}$$

where we use the boosted variables

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \tag{2.22}$$

In (2.21) we have allowed for time-dependence of the angle $\phi(t,s)$ with Ψt, taken into account the boundary condition (2.9) with Ωs where

$$\Omega = \frac{p}{2r} \tag{2.23}$$

and we now assume that the remaining functions $\varphi(u)$ and $z(u)$ are periodic,

$$\varphi(u + 2r\cosh\eta) = \varphi(u) \ , \ \ z(u + 2r\cosh\eta) = z(u) \tag{2.24}$$

This implies the identities $\int_{-r}^{r} ds\dot\varphi = 0 = \int_{-r}^{r} ds\varphi'$ which we shall use later. From now on, over-dot will denote $\frac{d}{du}$.

With the ansatz (2.21) the equations of motion (2.15) for ϕ becomes

$$\frac{d}{du}\left((1-z^2)\left(\dot\varphi + \Psi\sinh\eta + \Omega\cosh\eta\right)\right) = 0 \tag{2.25}$$

This equation implies that the quantity in front of the derivative is a constant, which we shall denote as j. Then

$$\dot\varphi = \frac{j}{1-z^2} - \Psi\sinh\eta - \Omega\cosh\eta \tag{2.26}$$

With the anzatz (2.21), the equations of motion (2.15) and (2.16) are second order differential equations for the functions $z(u)$ and $\varphi(u)$ of the variable u. Since they now

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have one variable, they are equivalent to the conservation of the energy-momentum tensor, i.e. the equations (2.20). For this reason, the Virasoro constraints (2.17) and (2.18) with the ansatz substituted are a first integral of the equations of motion. With the magnon ansatz (2.21), they are

$$\left(1-z^2\right)\left(\Psi-\Omega-e^{\eta}\dot{\varphi}\right)^2+e^{2\eta}\frac{\dot{z}^2}{1-z^2}=1 \tag{2.27}$$

$$\left(1-z^2\right)\left(\Psi+\Omega+e^{-\eta}\dot{\varphi}\right)^2+e^{-2\eta}\frac{\dot{z}^2}{1-z^2}=1 \tag{2.28}$$

These equations are compatible with (2.26) and give the equations for the parameters which determines j:

$$j=\frac{\sinh 2\eta}{2\left(\Psi\cosh\eta+\Omega\sinh\eta\right)} \tag{2.29}$$

and the equations which determines $\dot{z}(u)$:

$$\left(\frac{dz}{du}\right)^2=\frac{\left(z^2-z_{\text{min}}^2\right)\left(z_{\text{max}}^2-z^2\right)}{z_{\text{max}}^2-z_{\text{min}}^2} \tag{2.30}$$

where the turning points are

$$z_{\text{max}}^2=1-\frac{\sinh^2\eta}{\left(\Psi\cosh\eta+\Omega\sinh\eta\right)^2} \tag{2.31}$$

$$z_{\text{min}}^2=1-\frac{\cosh^2\eta}{\left(\Psi\cosh\eta+\Omega\sinh\eta\right)^2} \tag{2.32}$$

These imply

$$\cosh\eta=\sqrt{\frac{1-z_{\text{min}}^2}{z_{\text{max}}^2-z_{\text{min}}^2}}\ ,\quad \sinh\eta=\sqrt{\frac{1-z_{\text{max}}^2}{z_{\text{max}}^2-z_{\text{min}}^2}} \tag{2.33}$$

$$\Psi\cosh\eta+\Omega\sinh\eta=\frac{1}{\sqrt{z_{\text{max}}^2-z_{\text{min}}^2}} \tag{2.34}$$

The solution is obtained by integrating (2.30),

$$u=-\int_{z_{\text{max}}}^{z(u)}dz\frac{\sqrt{z_{\text{max}}^2-z_{\text{min}}^2}}{\sqrt{z^2-z_{\text{min}}^2}\sqrt{z_{\text{max}}^2-z^2}}\ ,\qquad u>0 \tag{2.35}$$

$$u=-\int_{z(u)}^{z_{\text{max}}}dz\frac{\sqrt{z_{\text{max}}^2-z_{\text{min}}^2}}{\sqrt{z^2-z_{\text{min}}^2}\sqrt{z_{\text{max}}^2-z^2}}\ ,\qquad u<0 \tag{2.36}$$

We have chosen the constant of integration so that the maximum of $z(u)$, z_{max} occurs at $u=0$ and the minimum is at $u=\pm r\cosh\eta$. $\frac{dz}{du}$ is positive when $u<0$ and negative when $u>0$. The resulting solutions are even functions of u, $z(u)=z(-u)$. The result of the integrals in (2.35) are the incomplete elliptic integrals of the first kind.

$$u=\nu\int_0^{\hat{\theta}(z)}\frac{d\theta}{\sqrt{1-\nu^2\sin^2\theta}}=\nu F\left(\hat{\theta}(z),\nu\right)\ ,\qquad 0\leq u\leq r\cosh\eta \tag{2.37}$$

$$u = -\nu \int_0^{\hat{\theta}(z)} \frac{d\theta}{\sqrt{1-\nu^2 \sin^2\theta}} = -\nu F\left(\hat{\theta}(z), \nu\right) \quad , \qquad -r\cosh\eta \le u \le 0 \tag{2.38}$$

where

$$\hat{\theta}(z) = \arcsin\sqrt{\frac{z_{\mathrm{maz}}^2 - z^2}{z_{\mathrm{max}}^2 - z_{\mathrm{min}}^2}} \quad , \quad \nu = \sqrt{1 - \frac{z_{\mathrm{min}}^2}{z_{\mathrm{max}}^2}} \tag{2.39}$$

and we are using the standard notation for the arguments of the elliptic function given in ref. [21]. The argument of the function is $z(u)$ which is then given by a Jacobi elliptic function,

$$z(u) = z_{\mathrm{max}} \mathbf{dn}\left(\frac{u}{\nu}, \nu\right) \tag{2.40}$$

It is the finite size Giant Magnon solution, given in terms of two integration constants z_{max} and z_{min}. In the next subsection we will discuss how these constants can be determined in terms of the energy and angular momentum of the solution.

2.1 Constants of integration

We note that the length of the worldsheet is

$$\begin{aligned} r &= \int_0^r ds = -\frac{1}{\cosh\eta}\int_{z_{\mathrm{max}}}^{\mathrm{z_{min}}} dz \frac{du}{dz} \\ &= \frac{1}{\cosh\eta}\int_{z_{\mathrm{min}}}^{z_{\mathrm{max}}} dz \frac{\sqrt{z_{\mathrm{max}}^2 - z_{\mathrm{min}}^2}}{\sqrt{z_{\mathrm{max}}^2 - z^2}\sqrt{z^2 - z_{\mathrm{min}}^2}} = \frac{\nu}{\cosh\eta} K(\nu) \end{aligned} \tag{2.41}$$

where $K(\nu) = F(\frac{\pi}{2}, \nu)$ is the complete Elliptic integral of the first kind (see appendix. A). Remembering that $r = \frac{\Delta}{2\sqrt{2\lambda}}$, eq. (2.19), we see that this yields

$$\Delta = 2\sqrt{2\lambda}\left(\frac{z_{\mathrm{max}}^2 - z_{\mathrm{min}}^2}{z_{\mathrm{max}}\sqrt{1 - z_{\mathrm{min}}^2}} K(\nu)\right) \quad , \quad \nu = \sqrt{1 - \frac{z_{\mathrm{min}}^2}{z_{\mathrm{max}}^2}} \tag{2.42}$$

Also by relating the length of the worldsheet to $(z_{\mathrm{max}}, z_{\mathrm{min}})$, it ensures that the period of the inverse elliptic function in (2.40) is the correct one.

Next, we shall derive the equation for the world sheet momentum. In eq. (2.21), the zero-modes proportional to Ψ and Ω were separated so that the remaining function $\varphi(u)$ is periodic in u. This implies that $\int_{-r\cosh\eta}^{r\cosh\eta} du\dot{\varphi}(u) = 0$. Eq. (2.26) determines its derivative $\frac{d\varphi}{du}$ in terms of $z(u)$ and constants as $\frac{d}{du}\varphi(u) = \frac{j}{1-z^2} - (\Psi\sinh\eta + \Omega\cosh\eta)$. The right-hand-side of this equation is a periodic and even function of u. Then, integrating both sides over the range of the u's and using the above observation that the integral of the left-hand-side must vanish, we find the identity

$$j\int_0^{r\cosh\eta} du \frac{1}{1-z^2} = r\cosh\eta\,(\Psi\sinh\eta + \Omega\cosh\eta)$$

Using (2.26) and (2.42), and recalling (2.34) we find

$$\Psi\sinh\eta + \Omega\cosh\eta = \frac{1}{r}\frac{\sqrt{z_{\mathrm{max}}^2 - z_{\mathrm{min}}^2}}{z_{\mathrm{max}}\sqrt{1 - z_{\mathrm{max}}^2}} \Pi\left(\frac{z_{\mathrm{max}}^2 - z_{\mathrm{min}}^2}{z_{\mathrm{max}}^2 - 1}; \nu\right)$$

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$$\Psi \cosh\eta + \Omega \sinh\eta = \frac{1}{\sqrt{z_{\rm max}^2 - z_{\rm min}^2}} \tag{2.43}$$

which we can solve to get

$$\Omega = \frac{1}{r} \frac{\sqrt{1 - z_{\rm min}^2}}{z_{\rm max}\sqrt{1 - z_{\rm max}^2}} \Pi\left(\frac{z_{\rm max}^2 - z_{\rm min}^2}{z_{\rm max}^2 - 1}; \nu\right) - \frac{\sqrt{1 - z_{\rm max}^2}}{z_{\rm max}^2 - z_{\rm min}^2} \tag{2.44}$$

where Π is the complete elliptic integral of the third kind (see appendix. A).

Combining eqs. (2.23), (2.42) and (2.44) we find

$$\frac{p}{2} = \frac{\sqrt{1 - z_{\rm min}^2}}{z_{\rm max}\sqrt{1 - z_{\rm max}^2}} \left(\Pi\left(\frac{z_{\rm max}^2 - z_{\rm min}^2}{z_{\rm max}^2 - 1}; \nu\right) - \frac{1 - z_{\rm max}^2}{1 - z_{\rm min}^2} K(\nu)\right) \tag{2.45}$$

In the orbifold case discussed above, see eq.(2.12), we should just set $\frac{p}{2} = \frac{\pi m}{M}$, for an m-times wrapped string.

Finally we shall compute the angular momentum, which is given by the Noether charge

$$\begin{aligned} J \equiv \frac{J_1 - J_3}{2} &= -2\sqrt{2\lambda} \int_{-r}^{r} ds \left[\frac{X_1\dot{X}_2 - X_2\dot{X}_1}{2} - \frac{X_5\dot{X}_6 - X_6\dot{X}_5}{2}\right] = \sqrt{2\lambda}\int_{-r}^{r} ds \left(1 - z^2\right)\frac{d}{dt}\phi \\ &= 2\sqrt{2\lambda}\frac{\sqrt{z_{\rm max}^2 - z_{\rm min}^2}}{\cosh\eta} \int_{z_{\rm min}}^{z_{\rm max}} \frac{dz(1 - z^2)}{\sqrt{z_{\rm max}^2 - z^2}\sqrt{z^2 - z_{\rm min}^2}} (\Psi - \sinh\eta\dot{\varphi}) \end{aligned} \tag{2.46}$$

Then, we use (2.26), (2.34) and (A.1) to find the identity

$$J = 2\sqrt{2\lambda} z_{\rm max} \left(K(\nu) - E(\nu)\right) \tag{2.47}$$

where K, E and Π are the complete elliptic integrals of the first, second and third kinds, respectively (see appendix. A). Equations (2.42), (2.47) and (2.45) are identical, a part for the overall factors, to those quoted in eqs. (36), (37) and (38) of ref. [12] and, with minor misprints corrected and $a = 0$, (B.4), (B.5) and (B.6) of ref. [11]. In those works, they were found using a light-cone gauge, and in the latter the conformal gauge and the results for physical quantities agree with each other.

In principle, two of the equations (2.42), (2.47) and (2.45) can be used to determine $z_{\rm min}$ and $z_{\rm max}$ in terms of the target space quantities. The third then gives an equation for the spectrum of the magnon, relating Δ, J and p. In practice, this can be done in the limit where Δ and J are large. This limit will be discussed in the next section.

3. The magnon limit

The magnon limit takes Δ and J large, so that $\Delta - J$ remains finite. This is achieved by taking $z_{\rm min} \to 0$. Using eqs. (A.4), (A.5) and (A.6), we can find an asyptotic expansion of eqs. (2.42), (2.47) and (2.45),

$$\Delta = 2\sqrt{2\lambda} z_{\rm max} \left\{\ln\frac{4z_{\rm max}}{z_{\rm min}} + \frac{1}{4}\frac{z_{\rm min}^2}{z_{\rm max}^2}\left[(2z_{\rm max}^2 - 3)\ln\frac{4z_{\rm max}}{z_{\rm min}} - 1\right] + \cdots\right\} \tag{3.1}$$

$$J = 2\sqrt{2\lambda}z_{\rm max}\left\{-1+\left(1-\frac{1}{4}\frac{z_{\rm min}^2}{z_{\rm max}^2}\right)\ln\frac{4z_{\rm max}}{z_{\rm min}}+\cdots\right\} \tag{3.2}$$

$$\frac{p}{2} = \arcsin z_{\rm max} - \frac{1}{4}\frac{z_{\rm min}^2}{z_{\rm max}^2}z_{\rm max}\sqrt{1-z_{\rm max}^2}\left(2\ln\frac{4z_{\rm max}}{z_{\rm min}}+1\right)+\cdots \tag{3.3}$$

Then, in the leading order,

$$z_{\rm max} = \left|\sin\frac{p}{2}\right| + \cdots \tag{3.4}$$

$$z_{\rm min} = 4\left|\sin\frac{p}{2}\right| \ \exp\left(\frac{-\Delta}{2\sqrt{2\lambda}\left|\sin\frac{p}{2}\right|}\right)+\cdots\approx 0 \tag{3.5}$$

We have chosen the solution where $z(u)$ is a positive function and therefore $z_{\rm max}$ and $z_{\rm min}$ are positive numbers. The Giant Magnon achieves maximum height $z_{\rm max}$ which is itself maximal when $p=\pi$. The smallest value of $z(u)$, $z_{\rm min}$, is always smaller by a factor that is exponentially small in the size Δ and is zero in the Giant Magnon limit.

Taking the infinite J limit, we get the equation for the spectrum obtained in [8]

$$\Delta - J = 2\sqrt{2\lambda}\left|\sin\frac{p}{2}\right| + \cdots \tag{3.6}$$

From this equation we see that, for a very small magnon, $p\ll 1$, $\Delta - J\sim\sqrt{2\lambda}p$.

We see furthermore that, to the leading order (3.6), it is easy to find the explicit solution,

$$z(u) = \frac{\sin\frac{p}{2}}{\cosh u} \ \ , \ \ \phi(t,s) = t + \arctan\left(\tan\frac{p}{2}\tanh u\right) \tag{3.7}$$

which using the ansatz (2.7)–(2.8) gives back the infinite J limit solution found in [8].

Finally, the leading exponential corrections to the magnon limit are easy to find. To the next-to-leading order we compute

$$\Delta - J = 2\sqrt{2\lambda}\left\{\left|\sin\frac{p}{2}\right| - 4\left|\sin^3\frac{p}{2}\right|\exp\left(\frac{-\Delta}{\sqrt{2\lambda}\left|\sin\frac{p}{2}\right|}\right)+\cdots\right\} \tag{3.8}$$

The exponential correction is the leading finite-size correction to the Giant Magnon dispersion relation. For the orbifold (2.12), p in (3.8) should just be set to $p=\frac{2\pi m}{M}$, for an m-times wrapped string.

A. Complete elliptic integrals

Above we used the integral formulae for complete elliptic integrals of the first, second and third kinds, respectively

$$\int_{z_{\rm min}}^{z_{\rm max}} dz\frac{1}{\sqrt{z^2-z_{\rm min}^2}\sqrt{z_{\rm max}^2-z^2}} = \frac{1}{z_{\rm max}}K(\nu) \tag{A.1}$$

$$\int_{z_{\rm min}}^{z_{\rm max}} dz\frac{z^2}{\sqrt{z^2-z_{\rm min}^2}\sqrt{z_{\rm max}^2-z^2}} = z_{\rm max}E(\nu) \tag{A.2}$$

$$\int_{z_{\min}}^{z_{\max}} \frac{dz}{(1-z^2)\sqrt{z^2 - z_{\min}^2}\sqrt{z_{\max}^2 - z^2}} = \frac{\Pi\left(\frac{z_{\max}^2 - z_{\min}^2}{z_{\max}^2 - 1}; \nu\right)}{z_{\max}(1 - z_{\max}^2)} \tag{A.3}$$

where $\nu = \sqrt{1 - \frac{z_{\min}^2}{z_{\max}^2}}$. We have taken conventions for the arguments of these functions which are defined by ref. [21]. In the paper we used asymptotic expansions around the limit $z_{\min} \to 0$,

$$K(\nu) = \ln\left(4\frac{z_{\max}}{z_{\min}}\right) + \frac{1}{4}\frac{z_{\min}^2}{z_{\max}^2}\left(\ln\left(4\frac{z_{\max}}{z_{\min}}\right) - 1\right) + \cdots \tag{A.4}$$

$$E(\nu) = 1 + \frac{1}{4}\frac{z_{\min}^2}{z_{\max}^2}\left(2\ln\left(4\frac{z_{\max}}{z_{\min}}\right) - 1\right) + \cdots \tag{A.5}$$

$$\begin{aligned}\Pi\left(\frac{z_{\max}^2 - z_{\min}^2}{z_{\max}^2 - 1}; \nu\right) &= \left(1 - z_{\max}^2\right)\left[\ln\left(4\frac{z_{\max}}{z_{\min}}\right) + \frac{z_{\min}^2}{4z_{\max}^2}\left(\left(2z_{\max}^2 + 1\right)\ln\frac{4z_{\max}}{z_{\min}} - \left(z_{\max}^2 + 1\right)\right)\right] + \\ &\quad + \left(1 + \frac{z_{\min}^2}{2}\right) z_{\max}\sqrt{1 - z_{\max}^2}\, \arcsin z_{\max} + \cdots \end{aligned} \tag{A.6}$$

where the three dots indicate terms of order $z_{\min}^4 \ln z_{\min}$ in all cases.

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